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A Theorem for the Development of a Function as an Infinite Product.

By A. F. Carpenter.

The usual conception of change of any kind as being brought about by an additive process, is purely arbitrary. So far as quantitative relations are concerned, the multiplicative, or indeed, any one of the extensions—in the sense of De Morgan [*]—of the ordinary algebraic processes, may be substituted for the additive. Corresponding to each of these conceptions, a consistent calculus may be built up [†] involving theorems which, by analogy, should be extensions of, and indeed, should be reducible to, the theorems of the ordinary calculus. For example, the Quotiential Calculus, based upon the conception of variation as taking place through successive multiplication by a quantity $h \lesseqgtr 1$ and whose limit is 1, should produce a set of theorems analogous to those of the ordinary calculus, the exact relation being that the processes of addition, multiplication, ..., of the ordinary calculus are respectively replaced in the new theorems by multiplication, involution,..... As a specific instance, Taylor's Theorem should have, corresponding to it, in the Quotiential Calculus, a theorem whereby a function may be expanded into an infinite product. This is exactly the case, as the following development serves to illustrate.

I.

To conform with the new conception of variation, it will be well to restate the definition for continuity at a point.

Definition: *The function* $f(x)$ *is continuous for* $x = x_0$, *if* $\frac{f(x_0h)}{f(x_0)}$ *approaches the limit* 1 *as* h *approaches* 1 *through values either less than, or greater than,* 1.

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[*] Christine Ladd: "De Morgan's Extension of the Algebraic Processes," American Journal of Mathematics, Vol. III (1880).

[†] Moritz: "Generalization of the Differentiation Process," American Journal of Mathematics, Vol. XXIV, No. 3.

A function is continuous over an interval if it is continuous for every point in that interval. It is clear that the values $\pm\infty$ and 0 must be excluded from consideration, since for each, x_0h will not differ from x_0 and hence, every function would be continuous at these points. Any interval under consideration will then be finite and exclude the origin.

In addition to the definition of continuity, the proof of the principle theorem makes use of the following auxiliary theorem:

THEOREM I: *If $F(x)$ and $QF(x)$* are continuous in the closed interval (a, b), and $F(a) = F(b)$, then for some value x_1 of x, such that $a < x_1 < b$, will $QF(x_1) = 0$.*

Proof: If $F(x)$ is constant in the interval (a, b), then will $QF(x) = 0$ over the whole interval. If $F(x)$ is not constant, then will there be some value x_1 of x, $(a \leqq x_1 \leqq b)$ for which $F(x_1)$ is a maximum (or minimum). Suppose $F(x_1)$ is a maximum and let,

1st, $h > 1$;

then $\frac{F(x_1h)}{F(x_1)} \leqq 1$ and $\log_h \frac{F(x_1h)}{F(x_1)} \leqq 0$, hence $QF(x_1) = \lim\limits_{h=1} \log_h \frac{F(x_1h)}{F(x_1)} \leqq 0$:

2nd, $h < 1$;

then $\frac{F(x_1h)}{F(x_1)} \leqq 1$ and $\log_h \frac{F(x_1h)}{F(x_1)} \geqq 0$, hence $QF(x_1) = \lim\limits_{h=1} \log_h \frac{F(x_1h)}{F(x_1)} \geqq 0$:

therefore, $QF(x_1) = 0$.

A similar course of reasoning for $F(x_1)$, a minimum, will result in the same conclusion.

II.

Let $f(x)$ and $Qf(x)$ be continuous in the closed interval (a, b) and define a quantity P_1 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{P_1}} = 1, \qquad (a < b) \tag{1}$$

and write

$$F_1(x) = \frac{f(x)}{f(a)\left(\frac{x}{a}\right)^{P_1}}. \tag{2}$$

Then $F_1(b) = 1$, by (1), and $F_1(a) = 1$, by (2).†

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* $QF(x)$ = the first quotiential coefficient of $F(x)$ with respect to x. In like manner $Q^nF(x)$ = the nth quotiential coefficient.

† Neither $f(a) = 0$ nor $f(b) = 0$, else (1) would fail to define P_1.

Now* $QF_1(x) = Qf(x) - P_1$, whence $QF_1(x_1) = Qf(x_1) - P_1 = 0$, where x_1 is some value of x such that $a < x_1 < b$. (Theorem I.)

Then $P_1 = Qf(x_1)$, and (1) becomes

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(x_1)}} = 1. \tag{3}$$

Equation (3) may be said to express the law of the mean for Quotientiation.

To obtain a first extension of (3), define a quantity P_2 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+P_2\log\frac{b}{a}}} = 1, \tag{4}$$

and write

$$F_2(x) = \frac{f(x)}{f(a)\left(\frac{x}{a}\right)^{Qf(a)+P_2\log\frac{x}{a}}}. \tag{5}$$

Then $F_2(b) = 1$, by (4), and $F_2(a) = 1$, by (5).

Now

$$QF_2(x) = \phi(x) = Qf(x) - Qf(a) - 2P_2 \log\frac{x}{a}, \tag{6}$$

so that for some value x_1, of x, such that $a < x_1 < b$, $\phi(x_1) = 0$ (Theorem I); also $\phi(a) = 0$, by (6). Again,

$$Q\phi(x) = \frac{Qf(x)\,Q^2f(x) - 2P_2}{\phi(x)},$$

and for some value x_2, of x, such that $a < x_2 < x_1$,

$$Q\phi(x_2) = \frac{Qf(x_2)\,Q^2f(x_2) - 2P_2}{\phi(x_2)} = 0.$$

Hence,† $Qf(x_2)Q^2f(x_2) - 2P_2 = 0$, and $P_2 = \frac{1}{2!}\,Qf(x_2)Q^2f(x_2)$, so that (4) becomes

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+\frac{1}{2!}Qf(x_2)Q^2f(x_2)\log\frac{b}{a}}} = 1. \tag{7}$$

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*All the Quotientiation formulas involved in this discussion are easily developed without the use of Taylor's Theorem, or, indeed, without recourse to the differentiation process.

† $\phi(x_2)$ is not infinite, else at least one of the quantities $Qf(x_2)$, $Qf(a)$, $\log\frac{x_2}{a}$ would be infinite, by (6). But (a, b) is a finite interval excluding the origin, and by hypothesis, $Qf(x)$ is continuous.

A second extension of (3) is made by defining a quantity P_3 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+\frac{1}{2!}Qf(a)Q^2f(a)\log\frac{b}{a}+P_3\left(\log\frac{b}{a}\right)^2}}=1, \tag{8}$$

and proceeding as before, the resulting value of P_3 being

$$P_3=\frac{1}{3!}Qf(x_3)Q^2f(x_3)\left[Q^2f(x_3)+Q^3f(x_3)\right],$$

where x_3 is some value of x, such that $a < x_3 < b$.

By assuming the existence of the first n quotiential coefficients of $f(x)$, and writing ak for $b\left(k=\frac{b}{a}\right)$, the $(n-1)$th extension of (3) gives

$$\begin{aligned} f(ak) &= f(a)\,(k)^{C_1+C_2\log k+C_3(\log k)^2+\ldots.C_n(\log k)^{n-1}} \\ &= f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n}, \end{aligned}$$

where, by the nature of process used, it is clear that

$$\begin{aligned} C_1 &= Qf(a), \\ C_2 &= \tfrac{1}{2}C_1QC_1, \\ C_3 &= \tfrac{1}{3}C_2QC_2, \\ &\ldots\ldots\ldots\ldots, \\ C_n^* &= \tfrac{1}{n}C_{n-1}QC_{n-1}. \end{aligned}$$

If any coefficient, say C_n, proves constant when a (or x_n) is replaced by x, then will C_{n+1} and all succeeding coefficients be zero † and the expansion stops of itself. If this be not the case, then in order to find the proper value to assign to this development, it will be necessary to find an expression for the quotient

$$\frac{f(ak)}{f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n}},$$

assuming the existence of $f(x)$ and its first $n+1$ quotiential coefficients in the interval (a, ak). To do this, set

$$\frac{f(ak)}{f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n+P(\log k)^p}}=1, \tag{9}$$

---

* In C_n, a is replaced by x_n, where x_n is some value of x such that $a < x_n < b$.

† This is not equivalent to $Q^{n+1}f(x)=0$. The vanishing of any q. c. does not bring the development to an end, but leads only to a special case.

where p is any positive integer and P, a quantity defined by the equation itself, and consider the function

$$F(x)=\frac{f(ak)}{f(x)\,(e)^{C_1'\log\frac{ak}{x}+C_2'\left(\log\frac{ak}{x}\right)^2+C_3'\left(\log\frac{ak}{x}\right)^3+\ldots.C_n'\left(\log\frac{ak}{x}\right)^n+P\left(\log\frac{ak}{x}\right)^p}}, \tag{10}$$

where

$$\begin{aligned}C_1'&=Qf(x),\\ C_2'&=\tfrac{1}{2}C_1'QC_1',\\ C_3'&=\tfrac{1}{3}C_2'QC_2',\\ &\ldots\ldots\ldots\ldots,\\ C_n'&=\tfrac{1}{n}C_{n-1}'QC_{n-1}'.\end{aligned}$$

$F(a)=1$, by (9), and $F(ak)=1$, by (10), and upon quotientiating after a simple reduction,

$$QF(x)=Pp\left(\log\frac{ak}{x}\right)^{p-1}-C_n'QC_n'\left(\log\frac{ak}{x}\right)^n, \tag{11}$$

so that, by Theorem I,* $QF(x_0)=0$, where x_0 is some value of x such that $a<x_0<ak$. Replacing x by $x_0=ak^{\theta_1}$, where $\theta_1=$ a positive proper fraction, and solving the resulting equation for P, gives

$$P=\frac{1}{p}\overline{C}_nQ\overline{C}_n(\log k)^{n-p+1}\,\theta^{n-p+1}, \tag{12}$$

where $\overline{C}_n$ is the result of replacing x with ak^{θ_1} in C_n', and $\theta=1-\theta_1=$ a positive proper fraction.

By replacing P with its value from (12), equation (9) gives rise to the following:

Theorem II. *If $f(x)$ and its first $n+1$ quotiential coefficients are continuous in the interval (a, ak), then will*

$$f(ak)=f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n+P_n}, \tag{13}$$

where

$$P_n=\frac{1}{p}\overline{C}_nQ\overline{C}_n\theta^{n-p+1}(\log\mathrm{k})^{n+1},$$

p being any positive integer and θ a positive proper fraction. The values which suggest themselves for p are readily seen to be

$$\begin{aligned}p&=1,\qquad\text{whence } P_n=\overline{C}_nQ\overline{C}_n\theta^n(\log\mathrm{k})^{n+1};\\ p&=n+1,\text{ whence } P_n=\frac{1}{n+1}\overline{C}_nQ\overline{C}_n(\log k)^{n+1}.\end{aligned}$$

---

* $F(x)$ and $QF(x)$ are continuous in (a, ak) through the conditions imposed on $f(x)$.

It is also to be noted that if P_n approaches 0 as n approaches ∞, (13) may be written

$$f(ak)=f(a)\prod_{n=1}^{\infty}(e)^{C_n(\log k)^n}, \tag{14}$$

or, placing x for a,

$$f(xk)=f(x)\prod_{n=1}^{\infty}(e)^{C'_n(\log k)^n}. \tag{15}$$

Interchanging x and k in (15), gives

$$f(kx)=f(k)\prod_{n=1}^{\infty}(e)^{C_n(\log x)^n}, \tag{16}$$

and, when $k=1$, (16) becomes

$$f(x)=f(1)\prod_{n=1}^{\infty}(e)^{C_n(\log x)^n}. \tag{17}$$

Equations (15) and (16) at once suggest Taylor's Theorem, and (17), MacLaurin's Theorem.

III.

It will be observed that the coefficients $C_1, \ldots, C_n$, in (15), (16), (17), are not as readily expressed in terms of successive quotiential coefficients as are the coefficients in Taylor's Theorem, in terms of successive derivatives. But if the derivative notation be substituted for the quotiential, this difficulty is removed. For this reason, the extension of Theorem II, to functions of two or more independent variables, will be considered subsequent to the change of notation.

Now,

$$Qy=\frac{xdy}{ydx}=\frac{d\log y}{d\log x}=\frac{dy}{yd\log x},^* \tag{18}$$

so that

$$yQy=\frac{dy}{d\log x}. \tag{19}$$

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*Moritz, "Generalization of the Differentiation Process," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXIV, No. 3.

From (18) and (19) the coefficients $C_1, C_2, \ldots\ldots, C_n, \ldots\ldots$, may now be written

$$C_1 = \frac{d \log f(a)}{d \log a},$$

$$C_2 = \frac{1}{2\,!}\,\frac{d^2 \log f(a)}{(d \log a)^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$C_n = \frac{1}{n\,!}\,\frac{d^n \log f(a)}{(d \log a)^n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and equations (14) to (17) become

$$f(ak) = f(a) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\frac{d^n \log f(a)}{(d \log a)^n}(\log k)^n}, \tag{20}$$

$$f(xk) = f(x) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\frac{d^n \log f(x)}{(d \log x)^n}(\log k)^n}, \tag{21}$$

$$f(kx) = f(k) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\frac{d^n \log f(k)}{(d \log k)^n}(\log x)^n}, \tag{22}$$

$$f(x) \;\; = f(1) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\frac{d^n \log f(1)}{(d \log 1)^n}(\log x)^n}. \tag{23}$$

Consider now, the function

$$\phi(\gamma) = f(a^{\log \gamma} x,\; b^{\log \gamma} y) = f(\alpha, \beta), \tag{24}$$

where $\alpha = a^{\log \gamma} x$, $\beta = b^{\log \gamma} y$.

From (24) $\phi(1) = f(x, y)$; also

$$\gamma \frac{\partial \alpha}{\partial \gamma} = x a^{\log \gamma} \log a \text{ and } \gamma \frac{\partial \beta}{\partial \gamma} = y b^{\log \gamma} \log b. \tag{25}$$

Again

$$\frac{d\phi}{d\gamma} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \gamma} + \frac{\partial \phi}{\partial \beta} \cdot \frac{\partial \beta}{\partial \gamma}, \tag{26}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} = \frac{\partial \phi}{\partial \alpha} \cdot a^{\log \gamma}, \tag{27}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = \frac{\partial \phi}{\partial \beta} \cdot b^{\log \gamma}. \tag{28}$$

Whence, by (25), (26), (27) and (28),

$$\frac{d\log\phi}{d\log\gamma}=\frac{\gamma\dfrac{d\phi}{d\gamma}}{\phi}=\log a\frac{x\dfrac{\partial\phi}{\partial x}}{\phi}+\log b\frac{y\dfrac{\partial\phi}{\partial y}}{\phi}=\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y},$$

that is

$$\frac{d}{d\log\gamma}=\log a\frac{\partial}{\partial\log x}+\log b\frac{\partial}{\partial\log y}. \tag{29}$$

From (29)

$$\frac{d^2\log\phi}{(d\log\gamma)^2}=\frac{d}{d\log\gamma}\left(\frac{d\log\phi}{d\log\gamma}\right)=\left[\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y}\right]^{(2)},$$

and, in general

$$\frac{d^n\log\phi}{(d\log\gamma)^n}=\left[\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y}\right]^{(n)}. \tag{30}$$

From (30) and (23), since $\phi(1)=f(x,y)$, there results the equation

$$\phi(\gamma)=f(a^{\log\gamma}x,\ b^{\log\gamma}y)=f(x,y)\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\left[\log a\frac{\partial\log f(x,y)}{\partial\log x}+\log b\frac{\partial\log f(x,y)}{\partial\log y}\right]^{(n)}(\log\gamma)^n}. \tag{31}$$

When $\gamma=e$, (31) becomes

$$f(ax,by)=f(x,y)\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\left[\log a\frac{\partial\log f(x,y)}{\partial\log x}+\log b\frac{\partial\log f(x,y)}{\partial\log y}\right]^{(n)}} \tag{32}$$

The above extension to a function of two independent variables is general and will apply equally well to functions of any number of variables.

IV.

It remains to exhibit the exact nature of the relation between the theorems just established and certain theorems of the ordinary calculus.

Consider the function

$$F(y)=e^{f(\log y)}.$$

Applying (22) to $F(ky)$,

$$F(ky)=e^{f(\log ky)}=e^{f(k)}\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\frac{d^nf(\log k)}{(d\log k)^n}(\log y)^n}. \tag{33}$$

Now, writing $k=e^a$ and $y=e^x$, and taking the logarithm of each member, (33) reduces to

$$f(a+x)=f(a)+\sum_{n=1}^{\infty}\frac{x^n}{n!}f^{(n)}(a),$$

one of the common forms of Taylor's Series for a function of a single variable.

Again, consider the function

$$F(u, v, w, \ldots.) = e^{f(\log u, \log v, \log w, \ldots.)}.$$

Applying (32) to $F(gu, kv, lw, \ldots.)$,

$$\begin{aligned} F(gu, kv, lw, \ldots.) &= e^{f(\log gu, \log kv, \log lw, \ldots.)} \\ &= e^{f(\log u, \log v, \log w, \ldots.)} \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\left[\log g \frac{\partial f}{\partial \log u} + \log k \frac{\partial f}{\partial \log v} + \log l \frac{\partial f}{\partial \log w} + \ldots.\right]^{(n)}}. \qquad (34) \end{aligned}$$

Now writing

$$g = e^a,\ k = e^b,\ l = e^c, \ldots.,$$

$$u = e^x,\ v = e^y,\ w = e^z, \ldots.,$$

and taking the logarithm of each member, (34) reduces to

$$\begin{aligned} f(a+x, b+y, c+z, \ldots.) &= f(x, y, z, \ldots.) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!}\left[a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z} + \ldots.\right]^{(n)}, \end{aligned}$$

Taylor's Series for a function of any number of variables.

Or, again, writing

$$g = e^0,\ k = e^{ah},\ l = e^{2bh+ah^2}, \ldots.,$$

$$u = e^a,\ v = e^b,\ w = e^c, \ldots.,$$

and taking the logarithm of each member, (34) reduces to

$$\begin{aligned} & f(a, b+ah, c+2bh+ah^2, \ldots.) \\ &= f(a, b, c, \ldots.) + \sum_{n=1}^{\infty} \frac{1}{n!}\left[ah\frac{\partial}{\partial b} + h(2b+ah)\frac{\partial}{\partial c}\right. \\ &\qquad\qquad \left. + h(3c+3bh+ah^2)\frac{\partial}{\partial d} + \ldots.\right]^n f \\ &= f(a, b, c, \ldots.) + \sum_{n=1}^{\infty} \frac{1}{n!}\left[h\left(a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c} + 3c\frac{\partial}{\partial d} + \ldots.\right)\right. \\ &\qquad \left. + h^2\left(a\frac{\partial}{\partial c} + 3b\frac{\partial}{\partial d} + \ldots.\right) + h^3\left(a\frac{\partial}{\partial d} + 4b\frac{\partial}{\partial e} + \ldots.\right)\right]^n f, \qquad (35) \end{aligned}$$

where it is to be noted that the nth power of the operator is no longer equivalent to n successive applications of its first power. If, following Sylvester,* we write

$$
\begin{aligned}
&f(a,\, b+ah,\, c+2bh+ah^2,\, \ldots\ldots) = f_1,\\
&a\frac{\partial}{\partial b}+2b\frac{\partial}{\partial c}+3c\frac{\partial}{\partial d}+\ldots\ldots = \Omega\,,\\
&a\frac{\partial}{\partial c}+3b\frac{\partial}{\partial d}+\ldots\ldots\ldots\ldots = \Omega_1,\\
&a\frac{\partial}{\partial d}+4b\frac{\partial}{\partial e}+\ldots\ldots\ldots\ldots = \Omega_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
$$

(35) becomes

$$f_1 = f + \sum_{n=1}^{\infty}\frac{1}{n!}\left(h\Omega + h^2\Omega_1 + h^3\Omega_2 + \ldots\ldots\right)^n,$$

or, more concisely,

$$f_1 = e^{h\Omega + h^2\Omega_1 + h^3\Omega_2 + \ldots\ldots} f. \tag{36}$$

Equation (36) is one form of what Sylvester chooses to call Salmon's Theorem.

V.

Theorem II can easily be established for functions of complex variables, the work necessitating, of course, the defining of quotientiation and its inverse, for such functions, and involving also the proof of a theorem expressed by the equation

$$f(a) = \left[\Pi e^{\frac{\operatorname{Log} f(z)}{\operatorname{Log} z/a}(qz-1)}\right]^{\frac{1}{2\Pi i}}.\dagger$$

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* Sylvester, *Mathematical Papers*, Vol. III, pp. 88–92.

† The inverse of the quotientiation process is $Q^{-1}f(x) = \Pi e^{f(x)(qx-1)}$.